Adv Comput Math (2013) 39:273–288
DOI 10.1007/s10444-012-9279-7

Computing some classes of Cauchy type singular integrals with *Mathematica* software

**Ana C. Conceição · Viktor G. Kravchenko ·
José C. Pereira**

Received: 25 October 2011 / Accepted: 9 August 2012 /
Published online: 4 September 2012

Abstract We constructed an algorithm, [SInt], for computing some classes of Cauchy type singular integrals on the unit circle. The design of [SInt] was focused on the possibility of implementing on a computer all the extensive symbolic and numeric calculations present in the algorithm. Furthermore, we show how the factorization algorithm described in Conceição et al. (2010) allowed us to construct and implement the [SIntAFact] algorithm for calculating several interesting singular integrals that cannot be computed by [SInt]. All the above techniques were implemented using the symbolic computation capabilities of the computer algebra system *Mathematica*. The corresponding source code of [SInt] is made available in this paper. Several examples of nontrivial singular integrals computed with both algorithms are presented.

Communicated by: Zhongying Chen.

Electronic supplementary material The online version of this article (doi:10.1007/s10444-012-9279-7) contains supplementary material, which is available to authorized users.

This research was partially supported by CEAF at Instituto Superior Técnico (Portugal).

A. C. Conceição (✉) · V. G. Kravchenko
CEAF, CEDMES, Departamento de Matemática, Faculdade de Ciências e Tecnologia,
Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal
e-mail: aicdoisg@gmail.com

J. C. Pereira
CEAF, CEDMES, Departamento de Engenharia Electrónica e Informática,
Faculdade de Ciências e Tecnologia, Universidade do Algarve,
Campus de Gambelas, 8005-139 Faro, Portugal
e-mail: unidadeimaginaria@gmail.com

Keywords Singular integrals · Symbolic computation · *Mathematica* software system · Factorization theory

Mathematics Subject Classifications (2010) Primary 30E20 · 68W30 · 47A68; Secondary 45E05 · 47A10

1 Introduction

In recent years, several software applications were made available to the general public with extensive capabilities of symbolic computation. These applications, known as computer algebra systems (CAS), allow to delegate to a computer all, or a significant part, of the symbolic and numeric calculations present in many mathematical algorithms. In our work we use the computer algebra system *Mathematica* (http://www.wolfram.com) to implement for the first time on a computer analytical algorithms developed by us and others within the Operator Theory.

The main goal of this paper is to present the analytical algorithms [SInt] and [SIntAFact] that compute some classes of Cauchy type singular integrals on the unit circle. Both algorithms were implemented using the symbolic computation capabilities of the computer algebra system *Mathematica*.

Singular integrals are classic mathematical objects with a vast array of applications in the main scientific research areas (see, for instance, [1, 10, 12, 13, 16, 19]) and the importance of their study is globally acknowledged. There exist several numerical algorithms and approximation methods for evaluating some classes of singular integrals. Also, there are several analytical techniques that allow the exact computation of singular integrals for particular cases. However, there are no analytical algorithms, written and implemented (up to our knowledge), for computing singular integrals with general functions.

The implementation of the [SInt] algorithm with the *Mathematica* system makes the results of lengthy and complex calculations available in a simple way to researchers of different areas. Also, the automation of this mathematical process results in the creation of a previously non-existent functionality of symbolic computation. Thus, it is the computer algebra system that acquires new capabilities.

Currently, we are attempting to generalize [SInt] to other types of curves (namely the real line) and to other classes of singular integrals. We are also reusing some of its parts to construct new algorithms to solve integral equations, Riemann–Hilbert problems, and to obtain factorizations of some special matrix functions.

The source code of the [SInt] algorithm is available as a supplement of the online edition of this article.

Factorization Theory is closely related to the computation of singular integrals and its development is stimulated by the need to solve problems emerging from several fields in Mathematics and Physics (see, for instance, [2–4, 8, 9, 14, 15, 17]). Recently, we developed for the first time, and partially implemented on a computer, the generalized factorization algorithm [AFact] for special classes of rational and non-rational matrix functions [5]. Due to its innovative character, the implementation of [AFact] potentiates the future design of algorithms dedicated to specific domains of application [6].

In this paper we describe the [SIntAFact] algorithm which uses [AFact] to compute Cauchy type singular integrals as a by-product of the factorization of a matrix function. The singular integrals computed by [SIntAFact] in general cannot be computed by the [SInt] algorithm. Also, as discussed in Sections 6 and 7, the [SIntAFact] algorithm provides us with extra information about the class of inner functions. This information can be used to study the properties of this type of function.

The rest of the paper is organized as follows:

In Section 2 we introduce the notion of Cauchy type singular integral and describe the calculation techniques for computing some classes of this kind of integral.

Sections 3 and 6 are dedicated to the formal description of the [SInt] and [SIntAFact] algorithms, respectively.

In Section 5 we describe how the implementation of the factorization algorithm [AFact], presented in [5], allows us to construct and implement the [SIntAFact] algorithm for calculating several interesting singular integrals that cannot be computed by [SInt].

In Sections 4 and 7, we present several examples of nontrivial singular integrals obtained with the algorithms [SInt] and [SIntAFact], respectively.

2 Calculation techniques for computing Cauchy type singular integral

In this section we describe some new calculation techniques for computing Cauchy type singular integrals, on the unit circle $\mathbb{T}$, with an integrand factor that can be represented as

$$\varphi(t) = r(t)[x_+(t) + y_-(t)], \tag{2.1}$$

where x_+ and $\overline{y_-}$[1] are bounded analytic functions in the interior of the unit circle and r is a rational function without poles on $\mathbb{T}$.

These techniques were implemented on a computer with the *Mathematica* software system, thus automating the extensive symbolic and numeric calculations. The obtained algorithm is called [SInt] and is formally described in Section 3.

[1] The overline denotes the complex conjugate of y_- in the unit circle.

Let us consider the singular integral associated with the singular integral operator

$$S_{\mathbb{T}}\varphi(t) = \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t}\, d\tau,\ \ t \in \mathbb{T}, \tag{2.2}$$

with Cauchy kernel, defined on $L_2(\mathbb{T})$.

It is known that $S_{\mathbb{T}}$ is a selfadjoint, unitary, and bounded operator in $L_2(\mathbb{T})$ (see, for instance, [10, 13]). Thus, we can associate with this operator two projection operators

$$P_{\pm} = (I \pm S_{\mathbb{T}})/2, \tag{2.3}$$

where I represents the identity operator.

Obviously, we have that

$$S_{\mathbb{T}} = 2P_{+} - I \tag{2.4}$$

Let us consider $H_\infty(\mathbb{T})$, the class of all bounded and analytic functions in the interior of the unit circle and let $R(\mathbb{T})$ denote the algebra of rational functions without poles on $\mathbb{T}$.

The [SInt] algorithm computes (2.2) when we can represent the function $\varphi(t)$ as (2.1) where

$$x_{+},\ \overline{y_{-}} \in H_\infty(\mathbb{T}),\ \text{and}\ r \in R(\mathbb{T}).$$

The algorithm uses extensively the properties of the projection operators (2.3) that emerge when those operators are applied to functions in $H_\infty(\mathbb{T})$ (e.g. x_{+}) and in $\overline{H_\infty(\mathbb{T})}$, (e.g. y_{-}). [SInt] also explores the rationality of $r(t)$ for reducing all possible situations to a few basic cases.

After the decomposition of the rational function $r(t)$ in elementary fractions the singular integrals are computed using (2.4) and the following formulas:

$$P_{+}[x_{+}(t)(a+bt)^n] = b^n\left[x_{+}(t)\left(t+\frac{a}{b}\right)^n - \sum_{i=1}^{-n}\frac{x_{+}^{(i-1)}\left(-\frac{a}{b}\right)}{(i-1)!}\left(t+\frac{a}{b}\right)^{i+n-1}\right],$$
$$\left|\frac{a}{b}\right| < 1,\ \ n<0 \tag{2.5}$$

$$P_{+}\left(y_{-}(t)t^n\right) = \sum_{i=0}^{n}\frac{\overline{y_{+}^{(i)}(0)}}{i!}t^{n-i}, \qquad n \geq 0 \tag{2.6}$$

$$P_{+}\left[y_{-}(t)(a+bt)^n\right] = b^n\left[\sum_{i=1}^{-n}\frac{y_{-}^{(i-1)}\left(-\frac{a}{b}\right)}{(i-1)!}\left(t+\frac{a}{b}\right)^{n+i-1}\right],$$
$$\left|\frac{a}{b}\right| > 1,\ \ n<0 \tag{2.7}$$

where $x_{+},\ \overline{y_{-}} \in H_\infty(\mathbb{T});\ a,\ b \in \mathbb{C},\ b \neq 0;\ n \in \mathbb{Z}$, and we define $y_{+}(t) = \overline{y_{-}(t)}$ for any $t \in \mathbb{T}$.

Proof We provide proof only for formula (2.5). Formulas (2.6) and (2.7) can be demonstrated in a similar fashion.

For the sake of simplicity we define $k = -n > 0$. It follows that

$$
\begin{aligned}
P_+[x_+(t)(a+bt)^n] &= b^n P_+\left[\frac{x_+(t)}{\left(t+\frac{a}{b}\right)^k}\right] \\
&= b^n P_+\left[\left(\underbrace{\frac{x_+(t)-x_+\left(-\frac{a}{b}\right)}{t+\frac{a}{b}}}_{\equiv x_{1+}(t)} + \underbrace{\frac{x_+\left(-\frac{a}{b}\right)}{t+\frac{a}{b}}}_{(-)}\right)\underbrace{\frac{1}{\left(t+\frac{a}{b}\right)^{k-1}}}_{(-)}\right] \\
&= b^n P_+\left[\left(\underbrace{\frac{x_{1+}(t)-x_{1+}\left(-\frac{a}{b}\right)}{t+\frac{a}{b}}}_{\equiv x_{2+}(t)} + \underbrace{\frac{x_{1+}\left(-\frac{a}{b}\right)}{t+\frac{a}{b}}}_{(-)}\right)\underbrace{\frac{1}{\left(t+\frac{a}{b}\right)^{k-2}}}_{(-)}\right] \\
&\dots \\
&= b^n P_+\left[x_{k+}(t)\right] = b^n x_{k+}(t)
\end{aligned}
\tag{2.8}
$$

where $x_{k+}(t)$ is defined analogously to $x_{1+}(t)$ and $x_{2+}(t)$ after k steps.

On the other hand, since $|\frac{a}{b}| < 1$, in a neighborhood of $-\frac{a}{b}$, we can write

$$
\begin{aligned}
x_+(t) = x_+\left(-\frac{a}{b}\right) + x_+'\left(-\frac{a}{b}\right)\left(t+\frac{a}{b}\right) + \frac{x_+''\left(-\frac{a}{b}\right)}{2!}\left(t+\frac{a}{b}\right)^2 \\
+ \frac{x_+'''\left(-\frac{a}{b}\right)}{3!}\left(t+\frac{a}{b}\right)^3 + \dots
\end{aligned}
\tag{2.9}
$$

and, therefore, we have

$$
\begin{aligned}
x_{1+}(t) &= x_+'\left(-\frac{a}{b}\right) + \frac{x_+''\left(-\frac{a}{b}\right)}{2!}\left(t+\frac{a}{b}\right) + \frac{x_+'''\left(-\frac{a}{b}\right)}{3!}\left(t+\frac{a}{b}\right)^2 + \dots \\
x_{2+}(t) &= \frac{x_+''\left(-\frac{a}{b}\right)}{2!} + \frac{x_+'''\left(-\frac{a}{b}\right)}{3!}\left(t+\frac{a}{b}\right) + \dots \\
&\dots \\
x_{k+}(t) &= \frac{x_+^{(k)}\left(-\frac{a}{b}\right)}{k!} + \frac{x_+^{(k+1)}\left(-\frac{a}{b}\right)}{(k+1)!}\left(t+\frac{a}{b}\right) + \dots
\end{aligned}
\tag{2.10}
$$

Using (2.9) we can rewrite (2.10) as

$$x_{k+}(t) = \frac{x_+(t) - \sum_{i=1}^{k} \frac{x_+^{(i-1)}\left(-\frac{a}{b}\right)}{(i-1)!} \left(t + \frac{a}{b}\right)^{i-1}}{\left(t + \frac{a}{b}\right)^{k}} \tag{2.11}$$

and, from (2.11) and (2.8) we obtain (2.5). □

The [SInt] algorithm can be applied to particular functions $x_+(t)$ and $y_-(t)$ or it can compute the closed form of (2.2) as a general expression in $x_+(t)$ and $y_-(t)$.

There are three options to insert $r(t)$:

1. Input $r(t)$ directly.
2. Input the numerator, and the poles and multiplicities.
3. Input zeros, poles and multiplicities.

We note that, since the poles of $r(t)$ are a crucial information for this calculation technique, in the case of option 1, the success of the [SInt] algorithm is dependent on the possibility of finding those poles by solving a polynomial equation.

3 [SInt] algorithm

This section is dedicated to the formal description of the [SInt] algorithm.

In Fig. 1 we present the pseudo code for the [SInt] algorithm. The analysis of this code and of the corresponding flowchart, presented in Fig. 2, reveals that the crucial steps of the algorithm are the decomposition of $r(t)$ and the computation of projections. The calculations involved in these two steps can become quite lengthy as the expressions of the functions $r(t)$, $x_+(t)$, and $y_-(t)$ become larger and more complex. However, based on our experiments with the algorithm, it is reasonable to expect total execution times in the order of a few seconds for most inputs (see Section 4 for some examples of execution times for the [SInt] algorithm).

[SInt]

1 Enter the expressions for $r(t)$, $x_+(t)$, and $y_-(t)$. For $r(t)$ it is also possible just to provide the poles, zeros, and the corresponding algebraic multiplicities.

2 If necessary, construct the expression for $r(t)$. Define the relevant properties for the projection operators P_+ and P_-.

3 Decompose the rational function $r(t)$ in elementary fractions.

4 Define the auxiliary functions $X(t) = r(t)x_+(t)$ and $Y(t) = r(t)y_-(t)$.

5 Compute the projections $P_+X(t)$ and $P_+Y(t)$.

6 Compute the singular integrals $S_{\mathbb{T}}X(t)$ and $S_{\mathbb{T}}Y(t)$.

Fig. 1 Pseudo code for the [SInt] algorithm

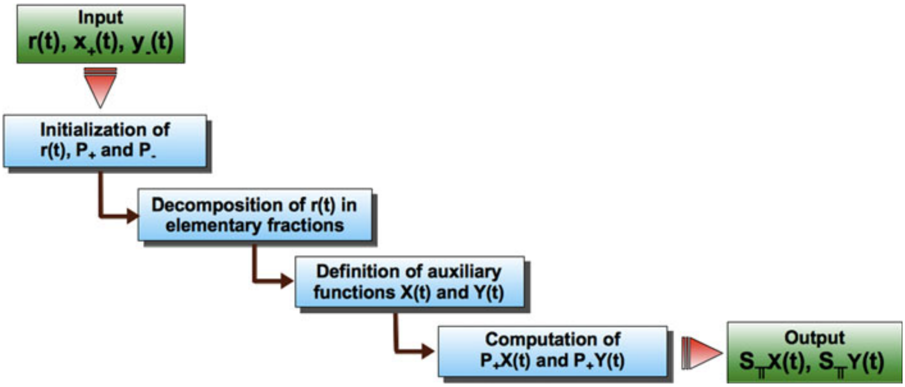

Fig. 2 Flowchart of [SInt] algorithm

As mentioned in Section 1, the [SInt] algorithm was implemented on a computer using the computer algebra system *Mathematica*. We note that, after running the algorithm, the singular integrals $S_{\mathbb{T}}X(t)$ and $S_{\mathbb{T}}Y(t)$ are defined as expressions in closed form and therefore, can be used in further calculations just like any other function in *Mathematica*.

The source code of the [SInt] algorithm is available as a supplement of the online edition of this article.

4 [SInt] examples

We will now present some examples of nontrivial singular integrals. For each input of functions $r(t)$, $x_+(t)$, and $y_-(t)$ the [SInt] algorithm computes the singular integrals $S_{\mathbb{T}}X(t)$ and $S_{\mathbb{T}}Y(t)$, where we define $X(t) = r(t)x_+(t)$ and $Y(t) = r(t)y_-(t)$.

All the examples were computed on a MacBook with a 2.4 GHz Intel Core 2 Duo processor and 2 GB of DDR3 RAM, running Mac OS X 10.6.8 (Snow Leopard) in single user mode.

Example 4.1 Let us consider the functions

$$r(t) = -\frac{1+t-5t^2+10t^4}{5(-4+t)^2t^3}; \qquad x_+(t) \equiv x_+(t); \qquad y_-(t) = t^{-k},\ k \geqslant 0$$

We obtain the singular integrals

$$S_{\mathbb{T}}X(t) = -\frac{128\left(1+t-5t^2+10t^4\right)x_+(t)+(-4+t)^2\left(\left(-16-24t+69t^2\right)x_+(0)-8t\left((2+3t)x'_+(0)+tx''_+(0)\right)\right)}{640(-4+t)^2t^3}$$

$$S_{\mathbb{T}}Y(t) = \frac{2^{-7-2k}t^{-3-k}\left(2^{7+2k}+2^{7+2k}t-5\,2^{7+2k}t^2+5\,4^{4+k}t^4+(576-9940k)t^{3+k}+(-2629+2485k)t^{4+k}\right)}{5(-4+t)^2}$$

Example 4.2 Let us consider the functions

$$r(t) = \frac{i + t^4}{t^2(-2i + t)}; \qquad x_+(t) = (-i + t)^{3i}; \qquad y_-(t) \equiv y_-(t)$$

We obtain the singular integrals

$$S_{\mathbb{T}} X(t) = \frac{-4i(-i)^{3i} + 12i(-i)^{3i}t - (6 + i)(-i)^{3i}t^2 + 2i(-i + t)^{3i} + 2t^4(-i + t)^{3i}}{2t^2(-2i + t)}$$

$$S_{\mathbb{T}} Y(t) = -\frac{-4t^2\left(4 + t^2\right)\overline{y_+(0)} - 4t^2(-2i + t)\overline{y'_+(0)} + (16 + i)t^2 y_-(2i) + 2iy_-(t) + 2t^4 y_-(t)}{2t^2(-2i + t)}$$

where we define, as previously, $y_+(t) = \overline{y_-(t)}$ for any $t \in \mathbb{T}$.

Example 4.3 Let us consider the functions

$$r(t) = \frac{-1 + t}{\left(-\frac{i}{2} + t\right)(-2i + t)^2}; \qquad x_+(t) \equiv x_+(t); \qquad y_-(t) \equiv y_-(t)$$

We obtain the singular integrals

$$S_{\mathbb{T}} X(t) = \frac{2\left((-8 + 4i)(-2i + t)^2 x_+\left(\frac{i}{2}\right) + 9(-1 + t)x_+(t)\right)}{9(-2i + t)^2(-i + 2t)}$$

$$S_{\mathbb{T}} Y(t) = \frac{2\left(-9(-1 + t)y_-(t) + 2(-i + 2t)\left(((8 + 7i) - (2 - i)t)y_-(2i) + (6 + 3i)(-2i + t)y'_-(2i)\right)\right)}{9(-2i + t)^2(-i + 2t)}$$

We note that the user may choose not to assign any particular expression to the input functions $x_+(t)$ and $y_-(t)$.[2] In this case, the obtained singular integrals are general functions of x_+ and/or y_-.

In Examples 4.1–4.3, the total execution times are in the order of a few tenths of a second. This indicates that even for more reasonably complex inputs the feasibility of the [SInt] algorithm will not be compromised by large execution times.

5 Computing singular integrals with the generalized factorization of a matrix function

In this section we describe how the implementation of the factorization algorithm [AFact], presented in [5], allowed us to construct and implement the [SIntAfact] algorithm for calculating several singular integrals that could not be computed by [SInt].

[2] The [SInt] algorithm always assumes that $x_+, \ \overline{y_-} \in H_\infty(\mathbb{T})$, and $r \in R(\mathbb{T})$ and therefore, its the user's responsibility to ensure that this is in fact the case.

Let us consider the special class of matrix functions (see, for instance, [3, 4, 7, 14, 15, 17])

$$A_\gamma(b) = \begin{pmatrix} 1 & b \\ \overline{b} & |b|^2 + \gamma \end{pmatrix} \tag{5.1}$$

where b is an essentially bounded function on the unit circle, that is, $b \in L_\infty(\mathbb{T})$, and γ is a non-zero complex constant.

The projection operators (2.3) allow us to decompose the space $L_2(\mathbb{T})$ in the topological direct sum

$$L_2(\mathbb{T}) = L_2^+(\mathbb{T}) \oplus L_2^{-,0}(\mathbb{T}),$$

where

$$L_2^+(\mathbb{T}) = \operatorname{Im} P_+, \;\; L_2^{-,0}(\mathbb{T}) = \operatorname{Im} P_-, \;\; L_2^-(\mathbb{T}) = L_2^{-,0}(\mathbb{T}) \oplus \mathbb{C}.$$

A matrix function of the class (5.1) admits a left canonical generalized factorization in $L_2(\mathbb{T})$ if it can be represented as

$$A_\gamma(b) = A_\gamma^+ A_\gamma^-,$$

where

$$(A_\gamma^+)^{\pm 1} \in \left[L_2^+(\mathbb{T})\right]_{2,2}, \;\; (A_\gamma^-)^{\pm 1} \in \left[L_2^-(\mathbb{T})\right]_{2,2},$$

and $A_+ P_+ A_+^{-1} I$ represents a bounded linear operator in $[L_2(\mathbb{T})]_2$.

A canonical generalized factorization of matrix functions of the type (5.1) has applications in several scientific research areas (see, for instance, [1, 8, 16]).

In [3, 5] and [7] we presented necessary and sufficient conditions for the existence of a canonical generalized factorization. Furthermore, we obtained expressions for an explicit canonical factorization of the matrix functions of the class (5.1).

Although we have theoretical results for $b \in L_\infty(\mathbb{T})$, we can always assume, without loss of generality, that $b \in H_\infty(\mathbb{T})$ (see, for instance, [3, 5, 7, 8]).

Let us consider the selfadjoint operator

$$N_+(b) = P_+ b\, P_- \overline{b}\, P_+, \;\; N_+(b): \; L_2(\mathbb{T}) \to L_2(\mathbb{T}). \tag{5.2}$$

and let $\rho\,(N_+(b))$ denote the resolvent set of the operator $N_+(b)$.

Obviously, if $-\gamma \in \rho(N_+(b))$, then the integral equations

$$(N_+(b) + \gamma I)\, u_+ = 1 \tag{5.3}$$

and

$$(N_+(b) + \gamma I)\, v_+ = b \tag{5.4}$$

are uniquely solvable.

We have the following result (see, for instance, [3, 5]).

Theorem 5.1 *$A_\gamma(b)$ admits a left canonical generalized factorization if and only if $-\gamma \in \rho(N_+(b))$. And, in that case,*

$$A_\gamma(b) = A_\gamma^+ A_\gamma^-, \tag{5.5}$$

where

$$A_\gamma^+ = \gamma \begin{pmatrix} u_+ & v_+ \\ P_+(\overline{b}u_+) & 1 + P_+(\overline{b}v_+) \end{pmatrix} \tag{5.6}$$

and

$$A_\gamma^- = \begin{pmatrix} 1 - P_-(\overline{b}v_+) & -P_-[b P_-(\overline{b}v_+)] \\ P_-(\overline{b}u_+) & 1 + P_-[b P_-(\overline{b}u_+)] \end{pmatrix}. \tag{5.7}$$

So, the factors A_γ^+ and A_γ^- can be represented using only the solutions u_+, v_+ of the non-homogeneous (5.3) and (5.4).

Let $H_{r,\theta}$ denote the set of all the functions of H_∞ that can be represented as the product of a rational outer function r and an inner function θ (i.e., θ is a bounded analytic function on the interior of the unit circle such that its modulus is equal to one a.e. on $\mathbb{T}$).

For the case when $b \in H_{r,\theta}$ and $-\gamma \in \rho(N_+(b))$ it is possible, by exploring the properties of the orthogonal projection $P_\theta = P_+ - \theta P_+ \overline{\theta} I$ [18] and the rationality of the outer function $r(t)$, to construct an algorithm (see [3, 5]) for solving some subclasses of integral equations of the type

$$(N_+(b) + \gamma I)\, u_+ = g_+. \tag{5.8}$$

In particular, it is now possible to solve the homogeneous equation ($g_+ \equiv 0$) and, therefore, we can know a priori if $-\gamma \in \rho(N_+(b))$. As a follow up, we constructed and partially implemented the [AFact] algorithm for $b \in H_{r,\theta}$ and $-\gamma \in \rho(N_+(b))$ [5]. In this case, we get the following representation for the projection entries of A_γ^+:

Corollary 5.2 *If $-\gamma \in \rho(N_+(b))$ and $b \in H_{r,\theta}$ then we can rewrite the second row of A_γ^+ as*

$$P_+(\overline{b}u_+) = \left[P_+(|r|^2 u_+) + \gamma u_+ - 1\right] b^{-1} \tag{5.9}$$

$$1 + P_+(\overline{b}v_+) = [P_+(|r|^2 v_+) + \gamma v_+] b^{-1} \tag{5.10}$$

Proof Under the hypothesis $-\gamma \in \rho(N_+(b))$ we know from Theorem 5.1 that exists the canonical generalized factorization (5.5).

Using (5.6), (5.7), and property $P_- = I - P_+$, and assuming that $b \in H_{r,\theta}$ we can rewrite A_γ^- as

$$A_\gamma^- = \begin{pmatrix} 1 - \overline{b}v_+ + P_+(\overline{b}v_+) & P_+(|r|^2 v_+) - |r|^2 v_+ \\ \overline{b}u_+ - P_+(\overline{b}u_+) & 1 + |r|^2 u_+ - P_+(|r|^2 u_+) \end{pmatrix} \tag{5.11}$$

Therefore, from (5.5) we get the relations

$$u_+ + u_+ P_+(\overline{b}v_+) - v_+ P_+(\overline{b}u_+) = \frac{1}{\gamma} \tag{5.12}$$

$$\overline{b}u_+ P_+(\overline{b}v_+) + \overline{b}u_+ - \overline{b}v_+ P_+(\overline{b}u_+) = \frac{1}{\gamma}\overline{b} \tag{5.13}$$

$$\begin{aligned} &P_+(|r|^2 v_+) P_+(\overline{b}u_+) - |r|^2 v_+ P_+(\overline{b}u_+) + |r|^2 u_+ - P_+(|r|^2 u_+) \\ &\quad + P_+(\overline{b}v_+) + |r|^2 u_+ P_+(\overline{b}v_+) - P_+(|r|^2 u_+) P_+(\overline{b}v_+) = \frac{1}{\gamma}|r|^2 \end{aligned} \tag{5.14}$$

Solving equalities (5.12), (5.13), and (5.14) for $P_+(\overline{b}u_+)$ and $P_+(\overline{b}v_+)$ we obtain formulas (5.9) and (5.10). □

One of the consequences of this new representation is that now we can obtain in a straightforward fashion the singular integrals $S_{\mathbb{T}}(\overline{b}u_+)$ and $S_{\mathbb{T}}(\overline{b}v_+)$, as the by-product of a canonical generalized factorization. This procedure is the [SIntAFact] algorithm that we describe in the next section. We note that the singular integrals obtained by [SIntAFact] cannot be computed, in general, by the [SInt] algorithm. One important example of this feature is when we consider θ to be a general inner function (see Examples 7.1 and 7.3).

6 [SIntAFact] algorithm

This section is dedicated to the formal description of the [SIntAFact] algorithm.

In Fig. 3 we present the pseudo code for the [SIntAFact] algorithm. According to Theorem 3.3 in [5] we have that

$$-\gamma \in \rho(N_+(b)) \Longleftrightarrow \kappa = 0$$

where κ is the dimension of the kernel of the operator $N_+(b) + \gamma I$.

Therefore, in step 2 the algorithm computes k to determine if a left canonical generalized factorization can be obtained, in accordance with Theorem 5.1. If that is the case, the [SIntAFact] obtains the singular integrals as a by-product of the generalized factorization (5.5). This means that, since the factorization factors cannot be known a priori, we cannot fully predict what singular integrals the [SIntAFact] algorithm will compute.

Similar to the [SInt] algorithm, it is possible not to assign any particular expression to the inner function θ and, in this case, the [SIntAFact] assumes that $-\gamma \in \rho(N_+(b))$, so that a canonical generalized factorization can be obtained and the corresponding singular integrals can be computed as functions of θ. Since $b \in H_{r,\theta}$ this assumption gives rise to some conditions that must be met by any particular inner function we may choose subsequently (see Examples 7.1 and 7.2 in Section 7).

[SIntAFact]

```
Enter the expressions for $r(t)$ and $\theta(t)$.  Enter the complex constant $\gamma$.
Solve the integral equation (5.8) for $g_+ \equiv 0$.
if $\kappa \neq 0$
    then if (no factorization)
            then
                The matrix function (5.1) does not admit a generalized
                factorization.  It is not possible to compute the singular
                integrals.
            else
                The matrix function (5.1) admits a non-canonical generalized
                factorization.  It is not possible yet to compute the
                singular integrals.  This step is under development.
    else
        Compute $u_+$, the solution of the integral equation (5.8) for $g_+ \equiv 1$.
        Compute $v_+$, the solution of the integral equation (5.8) for $g_+ \equiv r\theta$.
        Compute the factor $A_\gamma^+$.
        Compute the singular integrals $S_{\mathbb{T}}(\bar{b}u_+)$ and $S_{\mathbb{T}}(\bar{b}v_+)$.
```

Fig. 3 Pseudo code for the [SIntAFact] algorithm

On the other hand, according to Corollary 2.2 in [5], if $\gamma \in \mathbb{C} \setminus \mathbb{R}_0^-$ then $A_\gamma(b)$ admits a left canonical generalized factorization and, in this case, setting θ as a general expression, generates some conditions that *every* inner function must satisfy (see Example 7.3 in Section 7). Therefore, by fixing the value of γ and choosing several particular $\theta(t)$ functions and/or by fixing θ as a general expression and varying the value of γ over the set $\mathbb{C} \setminus \mathbb{R}_0^-$, these conditions can be studied for different rational functions $r(t)$ in order to generate hypothesis over more general properties of the inner functions themselves.

In summary, we recommend the following procedure for using the [SIntAFact] algorithm:

1. Choose a value for the constant γ and set the inner function θ as a general expression. The corresponding singular integrals are computed as functions of θ.
2. If $\gamma \in \mathbb{R}^-$ then particular singular integrals can only be obtained for those particular inner functions that satisfy some conditions, provided by the algorithm in step 1.
3. If $\gamma \in \mathbb{C} \setminus \mathbb{R}_0^-$ then $A_\gamma(b)$ admits a left canonical generalized factorization and the singular integrals can be obtained for any inner function. In addition, some conditions that must be satisfied by *every* inner function are obtained. These conditions can be studied by varying the value of γ over the set $\mathbb{C} \setminus \mathbb{R}_0^-$ and by choosing several particular $r(t)$ and $\theta(t)$ functions.[3]

[3]The [SIntAFact] algorithm always assumes that the function $r(t)$ is an outer function and that the given function $\theta(t)$ is an inner function and, therefore it is the user's responsibility to ensure that this is in fact the case for the chosen input.

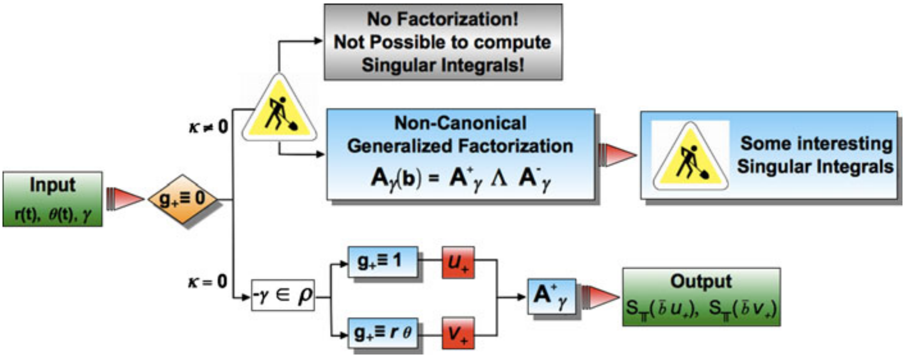

Fig. 4 Flowchart of [SIntAFact] algorithm

The flowchart for the [SIntAFact] algorithm is presented in Fig. 4.

As mentioned in Section 1, the [SIntAFact] algorithm was implemented on a computer using the computer algebra system *Mathematica*.[4]

7 [SIntAFact] examples

We will now present some examples of nontrivial singular integrals.

As before, all the following examples were computed in a MacBook with a 2.4 GHz Intel Core 2 Duo processor and 2 GB of DDR3 RAM, running Mac OS X 10.6.7 (Snow Leopard) in single user mode.

Example 7.1 Let us consider the functions and constant

$$r(t) = \frac{1}{-2+t}; \qquad \theta(t) \equiv \theta(t); \qquad \gamma = -\frac{1}{9}$$

For the case when $A_\gamma(b)$ admits a canonical generalized factorization we obtain the singular integrals

$$S_\mathbb{T}(\overline{b}u_+) = \frac{12(1+t)^2(-1+2t)\left(4+3\overline{\theta(0)}\left(-\theta(-1)+(1+t)\theta'(-1)\right)\right)-9t\bar{\theta}(t)(-3\overline{\theta(0)}g(t)+4h(t))}{4(-2+t)(1+t)^2(-1+2t)\theta'(-1)}$$

$$S_\mathbb{T}(\overline{b}\,v_+) = \frac{-4(1+t)^2(-1+2t)\left(\theta(-1)-3\theta'(-1)\right)+3t\bar{\theta}(t)g(t)}{2(-2+t)(1+t)^2(-1+2t)\theta'(-1)}$$

[4]We do not make available the corresponding source code in this paper because, the component responsible for computing the canonical generalized factorization of the matrix function $A_\gamma(b)$ is included in another paper by the same authors that is currently awaiting publication.

where

$$\overline{b}u_+ = \frac{9t\bar{\theta}(t)(-3\overline{\theta(0)}g(t)+4h(t))}{4(-2+t)(1+t)^2(-1+2t)\theta'(-1)}$$

$$\overline{b}v_+ = \frac{3\bar{\theta}(t)g(t)}{2\left(-2+\frac{1}{t}\right)(-2+t)(1+t)^2\theta'(-1)}$$

and

$$g(t) = (-2+t)^2\theta(-1)^2 - 3(-1+2t)\theta(t)\left(-\theta(-1)+(1+t)\theta'(-1)\right)$$

$$h(t) = (-3+6t)\theta(t) + (-2+t)^2\left(\theta(-1)+(1+t)\theta'(-1)\right)$$

In this example, the inner function θ must satisfy the condition[5] $\theta'(-1) \neq 0$, in order for the singular integrals to be well defined. For instance, we can choose the well known inner function $\theta(t) = e^{\frac{1+t}{-1+t}}$ (see, for instance, [11]) to obtain the following

Example 7.2 Let us consider the functions and constant

$$r(t) = \frac{1}{-2+t}; \qquad \theta(t) = e^{\frac{1+t}{-1+t}}; \qquad \gamma = -\frac{1}{9}$$

In this case $A_\gamma(b)$ admits a canonical generalized factorization and we obtain the singular integrals

$$S_{\mathbb{T}}(\overline{b}u_+) = \frac{3(9-8e+3t)(4+t(-1+4t))}{4e(-2+t)(1+t)^2} - \frac{9e^{-\frac{2t}{-1+t}}(3+2e(-1+t))(-2+t)t}{2(1+t)^2(-1+2t)}$$

$$S_{\mathbb{T}}(\overline{b}v_+) = \frac{10}{-2+t} - \frac{3t\left(-9+2e^{\frac{1+t}{1-t}}(-2+t)^2+15t+6t^2\right)}{2(-2+t)(1+t)^2(-1+2t)}$$

where

$$\overline{b}u_+ = \frac{27t(9-8e+3t)}{4e(-2+t)(1+t)^2} + \frac{9e^{-\frac{2t}{-1+t}}(3+2e(-1+t))(-2+t)t}{2(1+t)^2(-1+2t)}$$

$$\overline{b}v_+ = \frac{3t\left(-9+2e^{\frac{1+t}{1-t}}(-2+t)^2+15t+6t^2\right)}{2(-2+t)(1+t)^2(-1+2t)}$$

Example 7.3 Let us consider the functions and constant

$$r(t) = t-2i; \qquad \theta(t) \equiv \theta(t); \qquad \gamma = 1$$

[5]The necessary conditions for the existence of a canonical generalized factorization of $A_\gamma(b)$ are provided explicitly in the output of the [SIntAFact] algorithm. These conditions arise from the construction of an homogeneous linear system within the factorization of $A_\gamma(b)$, which we know to be uniquely solvable in the canonical case (see [5] for a more detailed explanation).

In this case, since we have chosen $\gamma \in \mathbb{C} \setminus \mathbb{R}_0^-$, Corollary 2.2 in [5] ensures that $A_\gamma(b)$ admits a canonical generalized factorization and we obtain the singular integrals

$$S_{\mathbb{T}}(\overline{b}u_+) = \frac{8\sqrt{5}t\bar{\theta}(z_1)}{2tf(\theta)} + \frac{(-i+2t)(-2i\sqrt{5}t(-2i+t)\theta(t)\bar{\theta}(z_1)-tf(\theta)+i(-1+\sqrt{5}+(1+\sqrt{5})\theta(z_2)\bar{\theta}(z_1)))\bar{\theta}(t)}{2t(-1-3it+t^2)f(\theta)}$$

$$S_{\mathbb{T}}(\overline{b}v_+) = 4it + \frac{10(1+\sqrt{5})+2(-7+3\sqrt{5})\theta(z_2)\bar{\theta}(z_1)4+2\sqrt{5}}{f(\theta)} - \frac{(1+2\mathrm{it})(2i\sqrt{5}\bar{\theta}(t)\theta(z_2)+t(-2i+t)(f(\theta)t-2i(f(\theta)-\theta(z_2)\bar{\theta}(z_1)+1)))}{t(-1-3it+t^2)f(\theta)}$$

where

$$\overline{b}u_+ = \frac{(1+2it)\left(1-\sqrt{5}-\theta(z_2)\bar{\theta}(z_1)-\sqrt{5}\theta(z_2)\bar{\theta}(z_1)+2\sqrt{5}t(-2i+t)\theta(t)\bar{\theta}(z_1)-itf(\theta)\right)\bar{\theta}(t)}{2t\left(-1-3it+t^2\right)f(\theta)}$$

$$\overline{b}v_+ = \frac{(1+2it)\left(2i\sqrt{5}\theta(z_2)\bar{\theta}(t)+t(-2i+t)\left(tf(\theta)-2i\left(f(\theta)-\theta(z_2)\bar{\theta}(z_1)+1\right)\right)\right)}{t\left(-1-3it+t^2\right)f(\theta)}$$

$$f(\theta) = 1+\sqrt{5}+\left(-1+\sqrt{5}\right)\theta(z_2)\bar{\theta}(z_1)$$

$$z_1 = \frac{1}{2}i\left(3+\sqrt{5}\right) \text{ and } z_2 = \frac{1}{2}i\left(3-\sqrt{5}\right)$$

References

1. Ablowitz, M.J., Clarkson, P.A.: Solitons, nonlinear evolution equations and inverse scattering. In: London Mathematical Society: Lecture Note Series, vol. 149. Cambridge University Press (1991)
2. Clancey, K., Gohberg, I.: Factorization of matrix functions and singular integral operators. In: Operator Theory: Advances and Applications, vol. 3. Birkhäuser, Basel (1981)
3. Conceição, A.C., Kravchenko, V.G.: About explicit factorization of some classes of non-rational matrix functions. Math. Nachr. **280**(9–10), 1022–1034 (2007)
4. Conceição, A.C., Kravchenko, V.G.: Factorization algorithm for some special matrix functions. In: Operator Theory: Advances and Applications, vol. 181, pp. 173–185. Birkhäuser (2008)
5. Conceição, A.C., Kravchenko, V.G., Pereira, J.C.: Factorization algorithm for some special non-rational matrix functions. In: Operator Theory: Advances and Applications, vol. 202, pp. 87–109. Birkhäuser (2010)
6. Conceição, A.C., Kravchenko, V.G., Pereira, J.C.: Explicit rational functions factorizations with symbolic computation. J. Symbol. Comput. (2012, to appear)

7. Conceição, A.C., Kravchenko, V.G., Teixeira, F.S.: Factorization of matrix functions and the resolvents of certain operators. In: Operator Theory: Advances and Applications, vol. 142, pp. 91–100. Birkhäuser (2003)
8. Conceição, A.C., Kravchenko, V.G., Teixeira, F.S.: Factorization of some classes of matrix functions and the resolvent of a Hankel operator. In: Factorization, Singular Operators and Related Problems, pp. 101–110. Kluwer Academic (2003)
9. Faddeev, L.D., Takhtayan, L.A.: Hamiltonian Methods in the Theory of Solitons. Springer, Berlin (1987)
10. Gobherg, I., Krupnik, N.: One-Dimensional Linear Singular Integral Equations. In: Operator Theory: Advances and Applications, vols. 53 and 54. Birkhäuser (1992)
11. Hoffman, K.: Banach Spaces of Analytic Functions. Dover, New York (1980)
12. Krupnik, N.Ya.: Banach Algebras with Symbol and Singular Integral Operators. In: Operator Theory: Advances and Applications, vol. 26. Birkhäuser (1987)
13. Kravchenko, V.G., Litvinchuk, G.S.: Introduction to the theory of singular integral operators with shift. In: Mathematics and its Applications, vol. 289. Kluwer Academic, Dordrecht (1994)
14. Kravchenko, V.G., Migdal'skii, A.I.: A regularization algorithm for some boundary-value problems of linear conjugation. Dokl. Math. **52**, 319–321 (1995)
15. Kravchenko, V.G., Nikolaichuk, A.M.: On partial indices of the Riemann problem for two pairs of functions. Sov. Math. Dokl. **15**, 438–442 (1974)
16. Litvinchuk, G.S.: Solvability theory of boundary value problems and singular integral equations with shift. In: Mathematics and its Applications, vol. 523. Kluwer Academic, Dordrecht (2000)
17. Litvinchuk, G.S., Spitkovskii, I.M.: Factorization of measurable matrix functions. In: Operator Theory: Advances and Applications, vol. 25. Birkhäuser, Basel (1987)
18. Nikol'skiï, N.K.: Treatise on the shift operator. spectral function theory. In: Grundlehren der Mathematischen Wissenschaften, vol. 273. Springer, Berlin (1986)
19. Prössdorf, S.: Some Classes of Singular Equations. North-Holland, Amsterdam (1978)